

POLYNOMIALS MEETING AX'S BOUND

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ABSTRACT. Let $f \in \mathbb{F}_q[X_1, \dots, X_n]$ with $\deg f = d > 0$ and let $Z(f) = \{(x_1, \dots, x_n) \in \mathbb{F}_q^n : f(x_1, \dots, x_n) = 0\}$. Ax's theorem states that $|Z(f)| \equiv 0 \pmod{q^{\lceil n/d \rceil - 1}}$, that is, $\nu_p(|Z(f)|) \geq m(\lceil n/d \rceil - 1)$, where $p = \text{char } \mathbb{F}_q$, $q = p^m$, and ν_p is the p -adic valuation. In this paper, we determine a condition on the coefficients of f that is necessary and sufficient for f to meet Ax's bound, that is, $\nu_p(|Z(f)|) = m(\lceil n/d \rceil - 1)$. Let $R_q(d, n)$ denote the q -ary Reed-Muller code $\{f \in \mathbb{F}_q[X_1, \dots, X_n] : \deg f \leq d, \deg_{X_j} f \leq q - 1, 1 \leq j \leq n\}$, and let $N_q(d, n; t)$ be the number of codewords of $R_q(d, n)$ with weight divisible by p^t . As applications of the aforementioned result, we find explicit formulas for $N_q(d, n; t)$ in the following cases: (i) $q = 2^m$, n even, $d = n/2$, $t = m + 1$; (ii) $q = 2$, $n/2 \leq d \leq n - 2$, $t = 2$; (iii) $q = 3^m$, $d = n$, $t = 1$; (iv) $q = 3$, $n \leq d \leq 2n$, $t = 1$.

1. INTRODUCTION

Let \mathbb{F}_q be the finite field with $q = p^m$ elements, where $p = \text{char } \mathbb{F}_q$. Let $f \in \mathbb{F}_q[X_1, \dots, X_n]$ with $\deg f = d > 0$ and let $Z(f) = \{(x_1, \dots, x_n) \in \mathbb{F}_q^n : f(x_1, \dots, x_n) = 0\}$. Ax's theorem [1] states that

$$(1.1) \quad \nu_p(|Z(f)|) \geq m \left(\left\lceil \frac{n}{d} \right\rceil - 1 \right),$$

where ν_p denotes the p -adic valuation. Ax's theorem is a strengthening of a result by Warning [21]. Further back along this line were a conjecture by Artin on the existence of nonzero roots of a homogeneous polynomial $f \in \mathbb{F}_q[X_1, \dots, X_n]$ with $n > \deg f$ and Chevalley's proof of Artin's conjecture; see [4].

The main ingredient of the original proof of Ax's theorem is the Stickelberger congruence of Gauss sums. A different proof based on the same idea but without using Gauss sums and the Stickelberger congruence was given by Ward [20].

Ax's theorem has been extended to several polynomials by N. Katz [10]. Assume that $f_i \in \mathbb{F}_q[X_1, \dots, X_n]$, $1 \leq i \leq r$, are such that $\deg f_i = d_i > 0$ and $d_1 = \max_{1 \leq i \leq r} d_i$, then

$$(1.2) \quad \nu_p(|Z(f_1) \cap \dots \cap Z(f_r)|) \geq m \left\lceil \frac{n - d_1 - \dots - d_r}{d_1} \right\rceil.$$

The original proof of Katz's theorem relied on sophisticated tools. A simpler proof was given by Wan [18, 19] using a method similar to Ax's. A more elementary proof of Katz's theorem for prime fields was found by Wilson [22]. Sun [17] further extended Katz's theorem for prime fields along the line of Wilson's approach.

Delsarte and McEliece [5] studied functions from a finite abelian group A to \mathbb{F}_q , where $\gcd(|A|, q) = 1$. Such functions were treated as elements of the group algebra

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$\mathbb{F}_q[A]$. Instead of polynomials in $\mathbb{F}_q[X_1, \dots, X_n]$ with a given degree, functions $f : A \rightarrow \mathbb{F}_q$ that belong to an ideal of $\mathbb{F}_q[A]$ were considered. (In coding theory, an ideal of $\mathbb{F}_q[A]$ is called an *abelian code*.) [5] established a lower bound for $\nu_p(|Z(f)|)$, which implies Ax's theorem when A is the cyclic group of order $q^n - 1$. D. Katz [9] generalized the result of [5] to a lower bound for $\nu_p(|Z(f_1) \cap \dots \cap Z(f_r)|)$, $f_1, \dots, f_r \in \mathbb{F}_q[A]$, and when A is the cyclic group of order $q^n - 1$, the generalized bound gives the theorem of N. Katz.

Although not obvious, (1.2) *actually follows from* (1.1), which was a finding by the author [7].

The bounds in (1.1) and (1.2) are both sharp; see [1, 10]. Therefore, improvements of these bounds are possible only under additional assumptions. For such improvements, see Cao [2], Cao and Sun [3], and O. Moreno and C. Moreno [14].

Focusing on (1.1), we note that another way to “improve” the bound is to find the next term in the p -adic expansion of $|Z(f)|$. In this paper, we will find an expression $E(f) \in \mathbb{F}_p$ such that

$$(1.3) \quad |Z(f)| \equiv q^{\lceil n/d \rceil - 1} E(f) \pmod{p^{m(\lceil n/d \rceil - 1) + 1}}.$$

Therefore,

$$\nu_p(|Z(f)|) \geq m \left(\left\lceil \frac{n}{d} \right\rceil - 1 \right) + 1$$

if and only if $E(f) = 0$. The expression $E(f)$ is a homogeneous polynomial over \mathbb{F}_p in the coefficients of f ; it is not explicit in general. However, in several special but nontrivial cases, $E(f)$ can be made explicit. By exploiting this fact, we obtain several explicit formulas for the number of codewords in a Reed-Muller code with weight divisible by a power of p . More precisely, let $R_q(d, n)$ denote the q -ary Reed-Muller code $\{f \in \mathbb{F}_q[X_1, \dots, X_n] : \deg f \leq d, \deg_{X_j} f \leq q - 1, 1 \leq j \leq n\}$, where \deg is the total degree and \deg_{X_j} is the degree in X_j , and let $N_q(d, n; t)$ be the number of codewords of $R_q(d, n)$ with weight divisible by p^t , where $p = \text{char } \mathbb{F}_q$. We find explicit formulas for $N_q(d, n; t)$ in the following cases: (i) $q = 2^m$, n even, $d = n/2$, $t = m + 1$; (ii) $q = 2$, $n/2 \leq d \leq n - 2$, $t = 2$; (iii) $q = 3^m$, $d = n$, $t = 1$; (iv) $q = 3$, $n \leq d \leq 2n$, $t = 1$.

In fact, for a finite abelian group A and $f \in \mathbb{F}_q[A]$, Delsarte and McEliece had found a formula for the next term in the p -adic expansion of $|Z(f)|$; see [5, (4.29)]. From that formula with $A = \mathbb{Z}/(q^n - 1)\mathbb{Z}$, one can derive a expression for the “next term” in Ax's theorem. The formula for the “next term” in [5], including the case $A = \mathbb{Z}/(q^n - 1)\mathbb{Z}$, involves the Fourier transform of f which takes values in an extension of \mathbb{F}_q . In comparison, the expression $E(f)$ determined in (2.22) of the present paper is considerably simpler.

In Section 2, we determine the expression $E(f)$ in (1.3). The method is a refinement of the original proof of Ax's theorem and relies on a careful analysis of the Stickelberger congruence of Gauss sums. Applications to Reed-Muller codes are discussed in Section 3.

Throughout the paper, for $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^n$, the relations $\mathbf{u} \equiv \mathbf{v} \pmod{k}$ and $\mathbf{u} \leq \mathbf{v}$ are meant to be component wise. We define

$$(1.4) \quad \Delta_n = \begin{bmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{bmatrix}_{n \times n}.$$

2. p -ADIC EXPANSION OF $|Z(f)|$

2.1. Gauss sum and Stickelberger congruence.

Facts gathered in this subsection can be found in any textbook on algebraic number theory, e.g., Lang [11, Ch. IV, §3].

For an integer $k > 0$, let $\zeta_k = e^{2\pi i/k}$. The ring of integers of a number field F is denoted by \mathfrak{o}_F . Let p be a rational prime, $m > 0$ and $q = p^m$. Let \mathfrak{p} be a prime of $\mathfrak{o}_{\mathbb{Q}(\zeta_{q-1})}$ lying above p . \mathfrak{p} is unramified over p and $\mathfrak{o}_{\mathbb{Q}(\zeta_{q-1})}/\mathfrak{p} = \mathbb{F}_q$. The Teichmüller set $T = \{0\} \cup \langle \zeta_{q-1} \rangle = \{0, \zeta_{q-1}^0, \dots, \zeta_{q-1}^{q-2}\}$ forms a system of coset representative of \mathfrak{p} in $\mathfrak{o}_{\mathbb{Q}(\zeta_{q-1})}$, that is, $\mathbb{F}_q = \mathfrak{o}_{\mathbb{Q}(\zeta_{q-1})}/\mathfrak{p} = \{t + \mathfrak{p} : t \in T\}$. The Teichmüller character $\chi_{\mathfrak{p}}$ is a multiplicative character of \mathbb{F}_q of order $q - 1$ defined by

$$\begin{aligned} \chi_{\mathfrak{p}} : \mathbb{F}_q = \mathfrak{o}_{\mathbb{Q}(\zeta_{q-1})}/\mathfrak{p} &\longrightarrow T \\ t + \mathfrak{p} &\longmapsto t, \quad t \in T. \end{aligned}$$

For each $a \in \mathbb{Z}$, the Gauss sum of $\chi_{\mathfrak{p}}^a$ is

$$g(\chi_{\mathfrak{p}}^a) = \sum_{t \in \langle \zeta_{q-1} \rangle} \chi_{\mathfrak{p}}^a(t) \zeta_p^{\text{Tr}_{q/p}(t+\mathfrak{p})} \in \mathfrak{o}_{\mathbb{Q}(\zeta_{p(q-1)})}.$$

Let \wp be the unique prime of $\mathfrak{o}_{\mathbb{Q}(\zeta_{p(q-1)})}$ lying above \mathfrak{p} . \wp is totally ramified over \mathfrak{p} with ramification index $e(\wp | \mathfrak{p}) = p - 1$.

For an integer $a \geq 0$ with base p expansion $a = a_0 + a_1p + \dots$, $0 \leq a_i \leq p - 1$, define $s(a) = a_0 + a_1 + \dots$ and $\gamma(a) = a_0!a_1!\dots$. The Stickelberger congruence states that for $1 \leq a \leq q - 2$,

$$(2.1) \quad \frac{g(\chi_{\mathfrak{p}}^{-a})}{(\zeta_p - 1)^{s(a)}} \equiv \frac{-1}{\gamma(a)} \pmod{\wp}.$$

2.2. p -adic expansion of $|Z(f)|$.

For $\mathbf{u} = (u_1, \dots, u_m) \in \mathbb{N}^n$, let $|\mathbf{u}| = u_1 + \dots + u_n$. If $\mathbf{x} = (x_1, \dots, x_n)$ is an n -tuple of elements from a commutative ring, we define $\mathbf{x}^{\mathbf{u}} = x_1^{u_1} \dots x_n^{u_n}$. Let $U_d = \{\mathbf{u} \in \mathbb{N}^n : |\mathbf{u}| \leq d\}$ and consider

$$f = \sum_{\mathbf{u} \in U_d} a_{\mathbf{u}} \mathbf{X}^{\mathbf{u}} \in \mathbb{F}_q[X_1, \dots, X_n],$$

where $\mathbf{X} = (X_1, \dots, X_n)$. We write $\sum_{\mathbf{u}}$ and $\prod_{\mathbf{u}}$ for $\sum_{\mathbf{u} \in U_d}$ and $\prod_{\mathbf{u} \in U_d}$, respectively. By [1, (5')], we have

$$(2.2) \quad q|Z(f)| = \sum_{i: U_d \rightarrow \{0, \dots, q-1\}} \left(\prod_{\mathbf{u}} \alpha_{\mathbf{u}}^{i(\mathbf{u})} \right) \left(\prod_{\mathbf{u}} c_{i(\mathbf{u})} \right) \sum_{\mathbf{t} \in T^{n+1}} \mathbf{t}^{\sum_{\mathbf{u}} i(\mathbf{u})(1, \mathbf{u})},$$

where $\alpha_{\mathbf{u}} \in T$ is such that

$$(2.3) \quad a_{\mathbf{u}} = \alpha_{\mathbf{u}} + \mathfrak{p},$$

and

$$(2.4) \quad c_i = \begin{cases} 1 & \text{if } i = 0, \\ -\frac{q}{q-1} & \text{if } i = q-1, \\ \frac{1}{q-1} g(\chi_{\mathfrak{p}}^{-i}) & \text{if } 0 < i < q-1. \end{cases}$$

By (2.1), we have $\nu_{\wp}(c_i) = s(i)$ for all $0 \leq i \leq q-1$. From the proof in [1, §3], we know that

$$(2.5) \quad \nu_{\wp} \left(\left(\prod_{\mathbf{u}} c_{i(\mathbf{u})} \right) \sum_{\mathbf{t} \in T^{n+1}} \mathbf{t}^{\sum_{\mathbf{u}} i(\mathbf{u})(1, \mathbf{u})} \right) \geq m(p-1) \left\lceil \frac{n}{d} \right\rceil$$

for all $i : U_d \rightarrow \{0, \dots, q-1\}$, where ν_{\wp} is the \wp -adic valuation. In fact, (2.5) implies (1.1) immediately. In what follows, we will reprove (2.5), and we will focus on those i for which the equal sign holds in (2.5).

When $\sum_{\mathbf{u}} i(\mathbf{u})(1, \mathbf{u}) \not\equiv (0, \dots, 0) \pmod{q-1}$,

$$\sum_{\mathbf{t} \in T^{n+1}} \mathbf{t}^{\sum_{\mathbf{u}} i(\mathbf{u})(1, \mathbf{u})} = 0.$$

When $\sum_{\mathbf{u}} i(\mathbf{u})(1, \mathbf{u}) = (0, \dots, 0)$,

$$\text{LSH of (2.5)} \geq \nu_{\wp}(q^{n+1}) = m(p-1)(n+1) > m(p-1) \left\lceil \frac{n}{d} \right\rceil.$$

Therefore, we assume that $\sum_{\mathbf{u}} i(\mathbf{u})(1, \mathbf{u}) \equiv (0, \dots, 0) \pmod{q-1}$ but $i \neq 0$ ($i(\mathbf{u}) \neq 0$ for at least one $\mathbf{u} \in U_d$). Let k be the number of nonzero components of $\sum_{\mathbf{u}} i(\mathbf{u})\mathbf{u}$. Then

$$(2.6) \quad \sum_{\mathbf{t} \in T^{n+1}} \mathbf{t}^{\sum_{\mathbf{u}} i(\mathbf{u})(1, \mathbf{u})} = (q-1)^{k+1} q^{n-k},$$

and

$$\begin{aligned} \text{LSH of (2.5)} &= \nu_{\wp} \left(\left(\prod_{\mathbf{u}} c_{i(\mathbf{u})} \right) (q-1)^{k+1} q^{n-k} \right) \\ &= \sum_{\mathbf{u}} s(i(\mathbf{u})) + m(p-1)(n-k) \\ (2.7) \quad &\geq m(p-1) \left\lceil \frac{k}{d} \right\rceil + m(p-1)(n-k) \\ &= m(p-1) \left(\left\lceil \frac{k}{d} \right\rceil + n - k \right) \\ (2.8) \quad &\geq m(p-1) \left\lceil \frac{n}{d} \right\rceil. \end{aligned}$$

In the above, inequality (2.8) is straightforward; inequality (2.7) was proved in [1] and will be explained below. First, we have

Fact 2.1. *When $d \geq 2$, the equal sign in (2.8) holds if and only if (i) $k = n$, or (ii) $k = n-1$ and $d \mid n-1$.*

Next, we determine the necessary and sufficient conditions for the equal sign to hold in (2.7). We have

$$d \sum_{\mathbf{u}} i(\mathbf{u}) \geq \sum_{\mathbf{u}} i(\mathbf{u})|\mathbf{u}| \geq k(q-1).$$

Since $\sum_{\mathbf{u}} i(\mathbf{u}) \equiv 0 \pmod{q-1}$, we have

$$(2.9) \quad \sum_{\mathbf{u}} i(\mathbf{u}) \geq (q-1) \left\lceil \frac{k}{d} \right\rceil.$$

For $a \in \{0, 1, \dots, q-1\}$ with base p expansion $a = a_0 + a_1p + \dots + a_{m-1}p^{m-1}$, $0 \leq a_j \leq p-1$, define

$$\tau(a) = a_{m-1} + a_0p + \dots + a_{m-2}p^{m-1}.$$

Then (2.9) remains true with $i(\mathbf{u})$ replaced by $\tau(i(\mathbf{u}))$. Therefore,

$$(2.10) \quad m(q-1) \left\lceil \frac{k}{d} \right\rceil \leq \sum_{h=0}^{m-1} \sum_{\mathbf{u}} \tau^h(i(\mathbf{u})) = \frac{q-1}{p-1} \sum_{\mathbf{u}} s(i(\mathbf{u})),$$

i.e.,

$$(2.5') \quad \sum_{\mathbf{u}} s(i(\mathbf{u})) \geq m(p-1) \left\lceil \frac{k}{d} \right\rceil,$$

which is the same as (2.7).

Fact 2.2. *The equal sign in (2.5') holds if and only if*

$$(2.11) \quad \sum_{\mathbf{u}} i(\mathbf{u})^{(j)} = (p-1) \left\lceil \frac{k}{d} \right\rceil \quad \text{for all } 0 \leq j \leq m-1,$$

where $(i(\mathbf{u})^{(0)}, \dots, i(\mathbf{u})^{(m-1)})$ are the base p digits of $i(\mathbf{u})$.

Proof. First note that the equal sign in (2.5') holds if and only if

$$(2.12) \quad \sum_{\mathbf{u}} \tau^h(i(\mathbf{u})) = (q-1) \left\lceil \frac{k}{d} \right\rceil \quad \text{for all } 0 \leq h \leq m-1.$$

We prove that (2.11) is equivalent to (2.12).

(\Rightarrow) Assume that (2.11) holds. Then for each $0 \leq h \leq m-1$ we have

$$\begin{aligned} \sum_{\mathbf{u}} \tau^h(i(\mathbf{u})) &= \sum_{\mathbf{u}} \tau^h \left(\sum_{j=0}^{m-1} i(\mathbf{u})^{(j)} p^j \right) = \sum_{\mathbf{u}} \sum_{j=0}^{m-1} i(\mathbf{u})^{(j)} \tau^h(p^j) \\ &= \sum_{j=0}^{m-1} \left(\sum_{\mathbf{u}} i(\mathbf{u})^{(j)} \right) \tau^h(p^j) = (p-1) \left\lceil \frac{k}{d} \right\rceil \sum_{j=0}^{m-1} \tau^h(p^j) \\ &= (p-1) \left\lceil \frac{k}{d} \right\rceil (1 + p + \dots + p^{m-1}) = (q-1) \left\lceil \frac{k}{d} \right\rceil. \end{aligned}$$

(\Leftarrow) Assume that (2.12) holds. Since

$$\begin{aligned} \tau^h(i(\mathbf{u})) &= \tau(\tau^{h-1}(i(\mathbf{u}))) = p\tau^{h-1}(i(\mathbf{u})) - (\tau^{h-1}(i(\mathbf{u})))^{(m-1)}(q-1) \\ &= p\tau^{h-1}(i(\mathbf{u})) - i(\mathbf{u})^{(m-h)}(q-1), \end{aligned}$$

where $m-h$ is taken modulo m , we have

$$\begin{aligned} (q-1) \left\lceil \frac{k}{d} \right\rceil &= \sum_{\mathbf{u}} \tau^h(i(\mathbf{u})) = \sum_{\mathbf{u}} (p\tau^{h-1}(i(\mathbf{u})) - i(\mathbf{u})^{(m-h)}(q-1)) \\ &= p(q-1) \left\lceil \frac{k}{d} \right\rceil - (q-1) \sum_{\mathbf{u}} i(\mathbf{u})^{(m-h)}, \end{aligned}$$

i.e.,

$$\sum_{\mathbf{u}} i(\mathbf{u})^{(m-h)} = (p-1) \left\lceil \frac{k}{d} \right\rceil.$$

□

We assume that $d \geq 2$ (to avoid trivial situations).

Definition 2.3. Let \mathcal{I} be the set of functions $i : U_d \rightarrow \{0, \dots, q-1\}$ such that

- (i) each component of $\sum_{\mathbf{u}} i(\mathbf{u})\mathbf{u}$ is a positive multiple of $q-1$;
- (ii) $\sum_{\mathbf{u}} i(\mathbf{u})^{(j)} = (p-1)\lceil n/d \rceil$ for all $0 \leq j \leq m-1$.

If $d \mid n-1$, let \mathcal{I}' be the set of functions $i : U_d \rightarrow \{0, \dots, q-1\}$ such that

- (i) one of the component of $\sum_{\mathbf{u}} i(\mathbf{u})\mathbf{u}$ is 0 and the other components are all positive multiples of $q-1$;
- (ii) $\sum_{\mathbf{u}} i(\mathbf{u})^{(j)} = (p-1)(n-1)/d$ for all $0 \leq j \leq m-1$.

If $d \nmid n-1$, define $\mathcal{I}' = \emptyset$.

By Facts 2.1 and 2.2, the equal sign in (2.5) holds if and only if $i \in \mathcal{I} \cup \mathcal{I}'$. Therefore by (2.2) and (2.6),

(2.13)

$$\begin{aligned} q|Z(f)| &\equiv \sum_{i \in \mathcal{I} \cup \mathcal{I}'} \left(\prod_{\mathbf{u}} \alpha_{\mathbf{u}}^{i(\mathbf{u})} \right) \left(\prod_{\mathbf{u}} c_{i(\mathbf{u})} \right) \sum_{\mathbf{t} \in T^{n+1}} \mathbf{t}^{\sum_{\mathbf{u}} i(\mathbf{u})(1, \mathbf{u})} \pmod{q^{\lceil n/d \rceil} \wp} \\ &= \sum_{i \in \mathcal{I}} \left(\prod_{\mathbf{u}} \alpha_{\mathbf{u}}^{i(\mathbf{u})} \right) \left(\prod_{\mathbf{u}} c_{i(\mathbf{u})} \right) (q-1)^{n+1} + \sum_{i \in \mathcal{I}'} \left(\prod_{\mathbf{u}} \alpha_{\mathbf{u}}^{i(\mathbf{u})} \right) \left(\prod_{\mathbf{u}} c_{i(\mathbf{u})} \right) (q-1)^n q. \end{aligned}$$

We know that

$$(2.14) \quad c_{i(\mathbf{u})} \equiv \frac{(\zeta_p - 1)^{s(i(\mathbf{u}))}}{\gamma(i(\mathbf{u}))} \pmod{(\zeta_p - 1)^{s(i(\mathbf{u}))} \wp}.$$

((2.14) is obvious when $i(\mathbf{u}) = 0$, and follows from (2.4) and (2.1) when $1 < i(\mathbf{u}) < q-1$. When $i(\mathbf{u}) = q-1$, (2.14) is easily verified directly.) Also note that

$$\begin{aligned} (2.15) \quad p &= \prod_{j=1}^{p-1} (\zeta_p^j - 1) = (\zeta_p - 1)^{p-1} \prod_{j=1}^{p-1} \frac{\zeta_p^j - 1}{\zeta_p - 1} \\ &\equiv (\zeta_p - 1)^{p-1} (p-1)! \pmod{(\zeta_p - 1)^p} \\ &\equiv -(\zeta_p - 1)^{p-1} \pmod{(\zeta_p - 1)^p}. \end{aligned}$$

Now combining (2.13) – (2.15) gives

(2.16)

$$\begin{aligned} q|Z(f)| &\equiv \sum_{i \in \mathcal{I}} \left(\prod_{\mathbf{u}} \alpha_{\mathbf{u}}^{i(\mathbf{u})} \right) \frac{(\zeta_p - 1)^{m(p-1)\lceil n/d \rceil}}{\prod_{\mathbf{u}} \gamma(i(\mathbf{u}))} (q-1)^{n+1} \\ &\quad + \sum_{i \in \mathcal{I}'} \left(\prod_{\mathbf{u}} \alpha_{\mathbf{u}}^{i(\mathbf{u})} \right) \frac{(\zeta_p - 1)^{m(p-1)(\lceil n/d \rceil - 1)}}{\prod_{\mathbf{u}} \gamma(i(\mathbf{u}))} (q-1)^n q \pmod{q^{\lceil n/d \rceil} \wp} \\ &\equiv q^{\lceil n/d \rceil} (-1)^{n+m\lceil n/d \rceil} \left[- \sum_{i \in \mathcal{I}} \prod_{\mathbf{u}} \frac{\alpha_{\mathbf{u}}^{i(\mathbf{u})}}{\gamma(i(\mathbf{u}))} + (-1)^m \sum_{i \in \mathcal{I}'} \prod_{\mathbf{u}} \frac{\alpha_{\mathbf{u}}^{i(\mathbf{u})}}{\gamma(i(\mathbf{u}))} \right] \\ &\quad \pmod{q^{\lceil n/d \rceil} \wp}. \end{aligned}$$

Let

$$(2.17) \quad \mathcal{E}(f) = (-1)^{n+m\lceil n/d \rceil} \left[- \sum_{i \in \mathcal{I}} \prod_{\mathbf{u}} \frac{\alpha_{\mathbf{u}}^{i(\mathbf{u})}}{\gamma(i(\mathbf{u}))} + (-1)^m \sum_{i \in \mathcal{I}'} \prod_{\mathbf{u}} \frac{\alpha_{\mathbf{u}}^{i(\mathbf{u})}}{\gamma(i(\mathbf{u}))} \right],$$

and write (2.16) as

$$(2.18) \quad |Z(f)| \equiv q^{\lceil n/d \rceil - 1} \mathcal{E}(f) \pmod{q^{\lceil n/d \rceil - 1} p}.$$

Since $\mathcal{E}(f) \in \mathbb{Q}(\zeta_{q-1})$, (2.18) gives

$$(2.19) \quad |Z(f)| \equiv q^{\lceil n/d \rceil - 1} \mathcal{E}(f) \pmod{q^{\lceil n/d \rceil - 1} p}.$$

Since $|Z(f)| \in \mathbb{Z}$, there exists $N \in \mathbb{Z}$ such that

$$(2.20) \quad \mathcal{E}(f) \equiv N \pmod{p}.$$

Taking images of both sides of (2.20) in $\{x \in \mathbb{Q}(\zeta_{q-1}) : \nu_{\mathfrak{p}}(x) \geq 0\}/\mathfrak{p} = \mathbb{F}_q$, we have

$$(2.21) \quad E(f) = N \pmod{p},$$

where

$$(2.22) \quad E(f) = (-1)^{n+m\lceil n/d \rceil} \left[- \sum_{i \in \mathcal{I}} \prod_{\mathbf{u}} \frac{a_{\mathbf{u}}^{i(\mathbf{u})}}{\gamma(i(\mathbf{u}))} + (-1)^m \sum_{i \in \mathcal{I}'} \prod_{\mathbf{u}} \frac{a_{\mathbf{u}}^{i(\mathbf{u})}}{\gamma(i(\mathbf{u}))} \right].$$

In fact, $E(f) \in \mathbb{F}_p$ because of (2.21).

To summarize, we have the following theorem.

Theorem 2.4. *Let $n \geq 1$, $d \geq 2$, and*

$$f = \sum_{\mathbf{u} \in U_d} a_{\mathbf{u}} \mathbf{X}^{\mathbf{u}} \in \mathbb{F}_q[X_1, \dots, X_n],$$

where $\mathbf{X} = (X_1, \dots, X_n)$. We have

$$(2.23) \quad |Z(f)| \equiv q^{\lceil n/d \rceil - 1} E(f) \pmod{q^{\lceil n/d \rceil - 1} p},$$

where $E(f)$ is given in (2.22). In particular, $\nu_p(|Z(f)|) \geq m(\lceil n/d \rceil - 1) + 1$ if and only if $E(f) = 0$.

Remark 2.5. $E(f)$ is a homogeneous polynomial of degree $(q-1)\lceil n/d \rceil$ over \mathbb{F}_p in the coefficients of f . In general, this expression is not explicit because \mathcal{I} and \mathcal{I}' are not. In the next section, we explore several special cases where $E(f)$ can be made explicit.

3. APPLICATIONS TO REED-MULLER CODES

3.1. Reed-Muller codes.

For a prime power $q = p^m$ and integers n, d with $n > 0$ and $0 \leq d \leq n(q-1)$, the q -ary Reed-Muller code $R_q(d, n)$ is defined as

$$(3.1) \quad R_q(d, n) = \{f \in \mathbb{F}_q[X_1, \dots, X_n] : \deg f \leq d, \deg_{X_j} f \leq q-1, 1 \leq j \leq n\}.$$

(For convenience, we define $R_q(-1, n) = \{0\}$.) It is known that [6, Result 1]

$$(3.2) \quad \dim_{\mathbb{F}_q} R_q(d, n) = \sum_{j \leq \lfloor d/q \rfloor} (-1)^j \binom{n}{j} \binom{d - qj + n}{n}.$$

For each $f \in R_q(d, n)$, its (Hemming) weight is $|f| = q^n - |Z(f)|$. The weight enumerator of $R_q(d, n)$ is not known except for the following special cases.

- (i) $d \leq 2$ or $d \geq n(q-1) - 3$. (For $d = 2$ and $q = 2$, see [13, Ch. 15, §2]; for $d = 2$ and q general, use the well known classification of quadratic forms over \mathbb{F}_q . For $d \geq n(q-1) - 3$, note that the dual of $R_q(d, n)$ is $R_q(d', n)$, where $d' = n(q-1) - 1 - d \leq 2$.)
- (ii) $q = 2$ and $n \leq 8$ ([8, 15]).

(iii) $q = 2, n = 9, d = 3$ ([16]).

For $t \geq 0$, let

$$N_q(d, n; t) = |\{f \in R_q(d, n) : \nu_p(|f|) \geq t\}|.$$

Ax's theorem implies that $N_q(d, n; t) = |R_q(d, n)|$ for $t \leq m(\lceil n/d \rceil - 1)$. We will use Theorem 2.4 to determine $N_q(d, n; t)$ with $t = m(\lceil n/d \rceil - 1) + 1$ in several cases; such formulas provide new information concerning the weight enumerators of the Reed-Muller codes involved. The cases we consider share a common assumption that $(p-1)\lceil n/d \rceil = 2$, that is, $p = 2$ and $\lceil n/d \rceil = 2$, or $p = 3$ and $\lceil n/d \rceil = 1$. Under this assumption, for each $i \in \mathcal{I}$ (Definition 2.3),

$$(3.3) \quad \sum_{\mathbf{u}} i(\mathbf{u})^{(j)} = 2 \quad \text{for all } 0 \leq j \leq m-1.$$

3.2. The case $q = 2^m$ and $d = n/2$.

Assume that $q = 2^m$, $n \geq 4$ is even, and $d = n/2$. Let $f = \sum_{\mathbf{u} \in U_{n/2}} a_{\mathbf{u}} \mathbf{X}^{\mathbf{u}} \in \mathbb{F}_q[X_1, \dots, X_n]$. Since $d \nmid n-1$, $\mathcal{I}' = \emptyset$ in Definition 2.3. Hence

$$(3.4) \quad E(f) = (-1)^{n+1} \sum_{i \in \mathcal{I}} \prod_{\mathbf{u}} \frac{a_{\mathbf{u}}^{i(\mathbf{u})}}{\gamma(i(\mathbf{u}))}.$$

If $i \in \mathcal{I}$, then

$$\sum_{\mathbf{u} \in U_d} i(\mathbf{u}) = \sum_{\mathbf{u}} \sum_{j=0}^{m-1} i(\mathbf{u})^{(j)} 2^j = 2 \sum_{j=0}^{m-1} 2^j = 2(q-1).$$

Since

$$n(q-1) \leq \sum_{\mathbf{u} \in U_{n/2}} i(\mathbf{u})|\mathbf{u}| \leq \frac{n}{2} \sum_{\mathbf{u} \in U_{n/2}} i(\mathbf{u}) = n(q-1),$$

we have $|\mathbf{u}| = n/2$ for all $\mathbf{u} \in U_{n/2}$ with $i(\mathbf{u}) > 0$ and we have

$$(3.5) \quad \sum_{|\mathbf{u}|=n/2} i(\mathbf{u})\mathbf{u} = (q-1, \dots, q-1).$$

Lemma 3.1. *$i \in \mathcal{I}$ if and only if there exist $\mathbf{u}_j, \mathbf{v}_j \in \{0, 1\}^n$, $0 \leq j \leq m-1$, with $|\mathbf{u}_j| = |\mathbf{v}_j| = n/2$, $\mathbf{u}_j + \mathbf{v}_j = (1, \dots, 1)$, such that for all $0 \leq j \leq m-1$,*

$$(3.6) \quad \begin{cases} i(\mathbf{u}_j)^{(j)} = i(\mathbf{v}_j)^{(j)} = 1, \\ i(\mathbf{u})^{(j)} = 0 \end{cases} \quad \text{if } \mathbf{u} \in U_{n/2} \setminus \{\mathbf{u}_j, \mathbf{v}_j\}.$$

Proof. (\Rightarrow) By Definition 2.3,

$$(3.7) \quad \sum_{|\mathbf{u}|=n/2} i(\mathbf{u})^{(j)} = 2 \quad \text{for all } 0 \leq j \leq m-1.$$

Choose $\mathbf{u}_{m-1}, \mathbf{v}_{m-1} \in U_{n/2}$ with $|\mathbf{u}_{m-1}| = |\mathbf{v}_{m-1}| = n/2$ such that $i(\mathbf{u}_{m-1})^{(m-1)} = i(\mathbf{v}_{m-1})^{(m-1)} = 1$. Since

$$\begin{aligned} (2^m - 1)(1, \dots, 1) &= \sum_{|\mathbf{u}|=n/2} i(\mathbf{u})\mathbf{u} \geq i(\mathbf{u}_{m-1})\mathbf{u}_{m-1} + i(\mathbf{v}_{m-1})\mathbf{v}_{m-1} \\ &\geq 2^{m-1}(\mathbf{u}_{m-1} + \mathbf{v}_{m-1}), \end{aligned}$$

it follows that $\mathbf{u}_{m-1} + \mathbf{v}_{m-1} \leq (1, \dots, 1)$, that is, $\mathbf{u}_{m-1}, \mathbf{v}_{m-1} \in \{0, 1\}^n$ and $\mathbf{u}_{m-1} + \mathbf{v}_{m-1} = (1, \dots, 1)$. For any $\mathbf{u} \in U_{n/2}$ with $|\mathbf{u}| = n/2$ and $\mathbf{u} \neq \mathbf{u}_{m-1}, \mathbf{v}_{m-1}$, we have $i(\mathbf{u})^{(m-1)} = 0$ by (3.7).

Now we have

$$\sum_{|\mathbf{u}|=n/2} \sum_{j=0}^{m-2} i(\mathbf{u})^{(j)} 2^j \mathbf{u} = (2^m - 1)(1, \dots, 1) - 2^{m-1}(1, \dots, 1) = (2^{m-1} - 1)(1, \dots, 1).$$

By the same argument, there exist $\mathbf{u}_{m-2}, \mathbf{v}_{m-2} \in \{0, 1\}^n$ with $|\mathbf{u}_{m-2}| = |\mathbf{v}_{m-2}| = n/2$ and $\mathbf{u}_{m-2} + \mathbf{v}_{m-2} = (1, \dots, 1)$ such that $i(\mathbf{u}_{m-2})^{(m-2)} = i(\mathbf{v}_{m-2})^{(m-2)} = 1$ and $i(\mathbf{u})^{(m-2)} = 0$ for all \mathbf{u} with $|\mathbf{u}| = n/2$ and $\mathbf{u} \neq \mathbf{u}_{m-2}, \mathbf{v}_{m-2}$. Continuing this way, we have $\mathbf{u}_j, \mathbf{v}_j$, $0 \leq j \leq m-1$, with the desired property.

(\Leftarrow) For each $0 \leq j \leq m-1$,

$$\sum_{\mathbf{u}} i(\mathbf{u})^{(j)} = i(\mathbf{u}_j)^{(j)} + i(\mathbf{v}_j)^{(j)} = 2 = (p-1)\lceil n/d \rceil.$$

Also,

$$\begin{aligned} \sum_{\mathbf{u}} i(\mathbf{u}) \mathbf{u} &= \sum_{\mathbf{u}} \left(\sum_{j=0}^{m-1} i(\mathbf{u})^{(j)} 2^j \right) \mathbf{u} = \sum_{j=0}^{m-1} 2^j (\mathbf{u}_j + \mathbf{v}_j) \\ &= \left(\sum_{j=0}^{m-1} 2^j \right) (1, \dots, 1) = (q-1)(1, \dots, 1). \end{aligned}$$

Hence $i \in \mathcal{I}$. □

It follows from Lemma 3.1 that

$$\begin{aligned} (3.8) \quad \sum_{i \in \mathcal{I}} \prod_{\mathbf{u}} a_{\mathbf{u}}^{i(\mathbf{u})} &= \sum_{\substack{\{\mathbf{u}_0, \mathbf{v}_0\}, \dots, \{\mathbf{u}_{m-1}, \mathbf{v}_{m-1}\} \\ \mathbf{u}_j, \mathbf{v}_j \in \{0, 1\}^n, |\mathbf{u}_j| = |\mathbf{v}_j| = n/2 \\ \mathbf{u}_j + \mathbf{v}_j = (1, \dots, 1)}} a_{\mathbf{u}_0} a_{\mathbf{v}_0} (a_{\mathbf{u}_1} a_{\mathbf{v}_1})^2 \cdots (a_{\mathbf{u}_{m-1}} a_{\mathbf{v}_{m-1}})^{2^{m-1}} \\ &= \left(\sum_{\substack{\{\mathbf{u}, \mathbf{v}\} \\ \mathbf{u}, \mathbf{v} \in \{0, 1\}^n, |\mathbf{u}| = |\mathbf{v}| = n/2 \\ \mathbf{u} + \mathbf{v} = (1, \dots, 1)}} a_{\mathbf{u}} a_{\mathbf{v}} \right)^{1+2+\dots+2^{m-1}}. \end{aligned}$$

Combining Theorem 2.4, (3.4) and (3.8) gives the following corollary.

Corollary 3.2. *Let $q = 2^m$ and $n \geq 4$ be even. Let*

$$f = \sum_{\mathbf{u} \in U_{n/2}} a_{\mathbf{u}} \mathbf{X}^{\mathbf{u}} \in \mathbb{F}_q[X_1, \dots, X_n].$$

Then $v_2(|Z(f)|) \geq m+1$ if and only if

$$(3.9) \quad \sum_{\substack{\{\mathbf{u}, \mathbf{v}\} \\ \mathbf{u}, \mathbf{v} \in \{0, 1\}^n, |\mathbf{u}| = |\mathbf{v}| = n/2 \\ \mathbf{u} + \mathbf{v} = (1, \dots, 1)}} a_{\mathbf{u}} a_{\mathbf{v}} = 0.$$

Replacing each $a_{\mathbf{u}}$ in (3.9) by an indeterminate $Y_{\mathbf{u}}$, we obtain a quadratic form

$$Q = \sum_{\substack{\{\mathbf{u}, \mathbf{v}\} \\ \mathbf{u}, \mathbf{v} \in \{0,1\}^n, |\mathbf{u}|=|\mathbf{v}|=n/2 \\ \mathbf{u}+\mathbf{v}=(1,\dots,1)}} Y_{\mathbf{u}} Y_{\mathbf{v}}$$

in $N = \binom{n}{n/2}$ indeterminates over \mathbb{F}_q . Order the indeterminates in a row $\mathbf{Y} = (Y_{\mathbf{u}} : \mathbf{u} \in \{0,1\}^n, |\mathbf{u}| = n/2)$ such that the indices \mathbf{u} and $\mathbf{u}^c := (1, \dots, 1) - \mathbf{u}$ appear in positions symmetric to the center of the row. Then

$$Q = \mathbf{Y} A \mathbf{Y}^t,$$

where

$$A = \begin{bmatrix} 0 & \Delta_{N/2} \\ 0 & 0 \end{bmatrix}_{N \times N}$$

and $\Delta_{N/2}$ is defined in (1.4). By [12, Theorem 6.32], the number of roots of Q in \mathbb{F}_q^N is

$$(3.10) \quad q^{N-1} + (q-1)q^{\frac{1}{2}N-1}.$$

Corollary 3.3. *Let $q = 2^m$ and $n \geq 4$ be even. Then*

$$(3.11) \quad N_q(n/2, n; m+1) = \left(q^{\binom{n}{n/2}-1} + (q-1)q^{\frac{1}{2}\binom{n}{n/2}-1} \right) q^{\dim_{\mathbb{F}_q} R_q(n/2, n) - \binom{n}{n/2}},$$

where

$$(3.12) \quad \dim_{\mathbb{F}_q} R_q(n/2, n) = \sum_{j \leq \lfloor n/2q \rfloor} (-1)^j \binom{n}{j} \binom{\frac{3n}{2} - qj}{n}.$$

Proof. (3.11) follows from Corollary 3.2 and (3.10); (3.12) follows from (3.2). \square

In the remaining three subsections, arguments and computations are similar to those in Subsection 3.2. Therefore, a fair amount of details is omitted.

3.3. The case $q = 2$ and $n/2 \leq d \leq n-2$.

Assume that $q = 2$, $n \geq 4$, and $n/2 \leq d \leq n-2$. Let $f = \sum_{\mathbf{u} \in U_d} a_{\mathbf{u}} \mathbf{X}^{\mathbf{u}} \in \mathbb{F}_2[X_1, \dots, X_n]$. Then $\mathcal{I}' = \emptyset$ and

$$E(f) = (-1)^{n+1} \sum_{i \in \mathcal{I}} \prod_{\mathbf{u}} \frac{a_{\mathbf{u}}^{i(\mathbf{u})}}{\gamma(i(\mathbf{u}))}.$$

Moreover, $i \in \mathcal{I}$ if and only if there exist $\mathbf{u}_0, \mathbf{v}_0 \in U_d \cap \{0,1\}^n$ with $\mathbf{u} + \mathbf{v} = (1, \dots, 1)$ such that

$$\begin{cases} i(\mathbf{u}_0) = i(\mathbf{v}_0) = 1, \\ i(\mathbf{u}) = 0 \end{cases} \quad \text{for all } \mathbf{u} \in U_d \setminus \{\mathbf{u}_0, \mathbf{v}_0\}.$$

Consequently,

$$\sum_{i \in \mathcal{I}} \prod_{\mathbf{u}} \frac{a_{\mathbf{u}}^{i(\mathbf{u})}}{\gamma(i(\mathbf{u}))} = \sum_{\substack{\{\mathbf{u}_0, \mathbf{v}_0\} \\ \mathbf{u}_0, \mathbf{v}_0 \in \{0,1\}^n, |\mathbf{u}_0|, |\mathbf{v}_0| \in [n-d, d] \\ \mathbf{u}_0 + \mathbf{v}_0 \geq (1, \dots, 1)}} a_{\mathbf{u}_0} a_{\mathbf{v}_0}.$$

Thus $\nu_2(|Z(f)|) \geq 2$ if and only if $(a_{\mathbf{u}} : \mathbf{u} \in \{0, 1\}^n, |\mathbf{u}| \in [n-d, d])$ is a root of the quadratic form

$$Q = \sum_{\substack{\{\mathbf{u}, \mathbf{v}\} \\ \mathbf{u}, \mathbf{v} \in \{0, 1\}^n, |\mathbf{u}|, |\mathbf{v}| \in [n-d, d] \\ \mathbf{u} + \mathbf{v} \geq (1, \dots, 1)}} Y_{\mathbf{u}} Y_{\mathbf{v}}.$$

Order the indeterminates of Q in a row $\mathbf{Y} = (Y_{\mathbf{u}} : \mathbf{u} \in \{0, 1\}^n, |\mathbf{u}| \in [n-d, d])$ such that $|\mathbf{u}|$ is increasing and the indices \mathbf{u} and $\mathbf{u}^c := (1, \dots, 1) - \mathbf{u}$ appear in positions symmetric to the center of the row. Then

$$Q = \mathbf{Y} A \mathbf{Y}^T,$$

where

$$A = \begin{bmatrix} & & & & & & & 1 \\ & & & & & & \cdot & * \\ & & & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot & \cdot \\ & & 1 & * & \cdot & \cdot & * & \\ * & * & * & * & \cdot & \cdot & * & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot & \cdot & * \\ & & & & \cdot & * & * & \\ & & & & & * & * & \end{bmatrix}_{N \times N}, \quad N = \sum_{j=n-d}^d \binom{n}{j}.$$

(The unmarked entries of A are all 0.) There exists $P \in \text{GL}(N, \mathbb{F}_2)$ such that

$$P A P^T = \begin{bmatrix} 0 & \Delta_{N/2} \\ 0 & 0 \end{bmatrix}.$$

Therefore the number of roots of Q in \mathbb{F}_2^N is $2^{N-1} + 2^{\frac{1}{2}N-1}$ [12, Theorem 6.32].

Corollary 3.4. *For $n \geq 4$ and $n/2 \leq d \leq n-2$,*

$$N_2(d, n; 2) = 2^{\binom{n}{0} + \dots + \binom{n}{d}-1} + 2^{2^{n-1}-1}.$$

3.4. The case $q = 3^m$ and $d = n$.

Assume that $q = 3^m$, $n \geq 2$, and $d = n$. Let $f = \sum_{\mathbf{u} \in U_n} a_{\mathbf{u}} \mathbf{X}^{\mathbf{u}} \in \mathbb{F}_q[X_1, \dots, X_n]$. Then $\mathcal{I}' = \emptyset$. Moreover, $i \in \mathcal{I}$ if and only if there exist $\mathbf{u}_j, \mathbf{v}_j \in \{0, 1, 2\}^n$, $0 \leq j \leq m-1$, with $|\mathbf{u}_j| = |\mathbf{v}_j| = n$ and $\mathbf{u}_j + \mathbf{v}_j = (2, \dots, 2)$ such that for all $0 \leq j \leq m-1$,

$$\begin{cases} i(\mathbf{u}_j)^{(j)} = i(\mathbf{v}_j)^{(j)} = 1 & \text{if } \mathbf{u}_j \neq \mathbf{v}_j, \\ i(\mathbf{u}_j)^{(j)} = 2 & \text{if } \mathbf{u}_j = \mathbf{v}_j, \\ i(\mathbf{u})^{(j)} = 0 & \text{if } \mathbf{u} \in U_n \setminus \{\mathbf{u}_j, \mathbf{v}_j\}. \end{cases}$$

We have

$$E(f) = (-1)^{n+m+1} \left(\sum_{\substack{\{\mathbf{u}, \mathbf{v}\} \\ \mathbf{u}, \mathbf{v} \in \{0, 1, 2\}^n, |\mathbf{u}| = |\mathbf{v}| = n \\ \mathbf{u} + \mathbf{v} = (2, \dots, 2)}} a_{\mathbf{u}} a_{\mathbf{v}} \right)^{1+3+\dots+3^{m-1}}.$$

Thus $\nu_3(|Z(f)|) \geq 1$ if and only if $(a_{\mathbf{u}} : \mathbf{u} \in \{0, 1, 2\}^n, |\mathbf{u}| = n)$ is a root of the quadratic form

$$Q = \sum_{\substack{\{\mathbf{u}, \mathbf{v}\} \\ \mathbf{u}, \mathbf{v} \in \{0, 1, 2\}^n, |\mathbf{u}| = |\mathbf{v}| = n \\ \mathbf{u} + \mathbf{v} = (2, \dots, 2)}} Y_{\mathbf{u}} Y_{\mathbf{v}}.$$

Order the indeterminates of Q in a row $\mathbf{Y} = (Y_{\mathbf{u}} : \mathbf{u} \in \{0, 1, 2\}^n, |\mathbf{u}| = n)$ such that the indices \mathbf{u} and $\mathbf{u}^c := (2, \dots, 2) - \mathbf{u}$ appear in positions symmetric to the center of the row. Then

$$Q = \mathbf{Y} A \mathbf{Y}^T,$$

where

$$A = \begin{bmatrix} 0 & \Delta_{(N+1)/2} \\ 0 & 0 \end{bmatrix}_{N \times N}, \quad N = \sum_{j \leq n/2} \binom{n}{j} \binom{n-j}{n-2j}.$$

The number of roots of Q in \mathbb{F}_q^N is q^{N-1} [12, Theorem 6.27].

Corollary 3.5. *Let $q = 3^m$ and $n \geq 2$. Then*

$$N_q(n, n; 1) = q^{\dim_{\mathbb{F}_q} R_q(n, n) - 1},$$

where

$$\dim_{\mathbb{F}_q} R_q(n, n) = \sum_{j \leq \lfloor n/q \rfloor} (-1)^j \binom{n}{j} \binom{2n - qj}{n}.$$

3.5. The case $q = 3$ and $n \leq d \leq 2n$.

Assume that $q = 3$, $n \geq 2$, and $n \leq d \leq 2n$. Let $f = \sum_{\mathbf{u} \in U_d} a_{\mathbf{u}} \mathbf{X}^{\mathbf{u}} \in \mathbb{F}_q[X_1, \dots, X_n]$. Then $\mathcal{I}' = \emptyset$. Moreover, $i \in \mathcal{I}$ if and only if there exist $\mathbf{u}_0, \mathbf{v}_0 \in \{0, 1, 2\}^n$ with $\mathbf{u}_0 \equiv \mathbf{v}_0 \pmod{2}$ and $\mathbf{u}_0 + \mathbf{v}_0 \geq (2, \dots, 2)$ such that

$$\begin{cases} i(\mathbf{u}_0) = i(\mathbf{v}_0) = 1 & \text{if } \mathbf{u}_0 \neq \mathbf{v}_0, \\ i(\mathbf{u}_0) = 2 & \text{if } \mathbf{u}_0 = \mathbf{v}_0, \\ i(\mathbf{u}) = 0 & \text{if } \mathbf{u} \in U_d \setminus \{\mathbf{u}_0, \mathbf{v}_0\}. \end{cases}$$

We have

$$E(f) = (-1)^{n+m+1} \sum_{\substack{\{\mathbf{u}, \mathbf{v}\} \\ \mathbf{u}, \mathbf{v} \in \{0, 1, 2\}^n, |\mathbf{u}|, |\mathbf{v}| \in [2n-d, d] \\ \mathbf{u} \equiv \mathbf{v} \pmod{2}, \mathbf{u} + \mathbf{v} \geq (2, \dots, 2)}} a_{\mathbf{u}} a_{\mathbf{v}}.$$

Thus $\nu_3(|Z(f)|) \geq 1$ if and only if $(a_{\mathbf{u}} : \mathbf{u} \in \{0, 1, 2\}^n, |\mathbf{u}| \in [2n-d, d])$ is a root of the quadratic form

$$Q = \sum_{\substack{\{\mathbf{u}, \mathbf{v}\} \\ \mathbf{u}, \mathbf{v} \in \{0, 1, 2\}^n, |\mathbf{u}|, |\mathbf{v}| \in [2n-d, d] \\ \mathbf{u} \equiv \mathbf{v} \pmod{2}, \mathbf{u} + \mathbf{v} \geq (2, \dots, 2)}} Y_{\mathbf{u}} Y_{\mathbf{v}}.$$

Order the indeterminates of Q in a row $\mathbf{Y} = (Y_{\mathbf{u}} : \mathbf{u} \in \{0, 1, 2\}^n, |\mathbf{u}| \in [2n-d, d])$ such that $|\mathbf{u}|$ is increasing and the indices \mathbf{u} and $\mathbf{u}^c := (2, \dots, 2) - \mathbf{u}$ appear in positions symmetric to the center of the row. Then

$$Q = \mathbf{Y} A \mathbf{Y}^T,$$

where

$$A = \begin{bmatrix} & & & & & & 1 \\ & & & & & \cdot & * \\ & & & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot & \cdot \\ & & 1 & * & \cdot & \cdot & * \\ & & & * & \cdot & \cdot & * \\ & & & & \cdot & \cdot & \cdot \\ & & & & & \cdot & \cdot \\ & & & & & & * \end{bmatrix}_{N \times N},$$

$$N = \sum_{j=2n-d}^d |\{\mathbf{u} \in \{0, 1, 2\}^n : |\mathbf{u}| = j\}|.$$

There exists $P \in \text{GL}(N, \mathbb{F}_3)$ such that

$$PAP^T = \begin{bmatrix} 0 & \Delta_{(N+1)/2} \\ 0 & 0 \end{bmatrix}.$$

Hence the number of roots of Q in \mathbb{F}_3^N is 3^{N-1} [12, Theorem 6.27].

Corollary 3.6. *Let $n \geq 2$ and $n \leq d \leq 2n$. Then*

$$N_3(d, n; 1) = 3^{\dim_{\mathbb{F}_3} R_3(d, n) - 1},$$

where

$$\dim_{\mathbb{F}_3} R_3(d, n) = \sum_{j \leq \lfloor d/3 \rfloor} (-1)^j \binom{n}{j} \binom{d - 3j + n}{n}.$$

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